Topology, Vol. 23, No. 3, pp. 271–276, 1984
Printed in the U.S.A.

0040–9383/84 $3.00 + .00

A NEW DIFFERENTIAL IN THE ADAMS SPECTRAL SEQUENCE

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(*Received* 1 *August* 1983)

§1. INTRODUCTION

IN A SERIES of papers [1,6,14,15], Barratt *et al.* used the Adams spectral sequence to determine the 2 component of $\pi_n S^0$ for $29 \leq n \leq 45$ together with a number of products and Toda brackets in this range. In this paper we show that there is an additional nonzero differential and determine its implications for products and Toda brackets.

The new differential follows from a general formula due to Makinen [7] relating differentials and Steenrod operations in the Adams spectral sequence of a sphere. It is interesting to note that Milgram[11] found a similar mistake in [6] (corrected in [1]) by means of a more limited formula of the same type.

We give Makinen's general formula in §2. We then collect some known calculations in the Adams spectral sequence for $\pi_* S^0$ in §3. Our new results are presented in §4, while §5 contains new proofs of results whose published proofs rely on the mistake we are correcting.

§2. THE GENERAL FORMULA

Let A be the mod 2 Steenrod algebra which operates on the cohomology of topological spaces. Let $\{E_r^{s,n}\}$ be the Adams spectral sequence

$$E_2^{s,n} = \mathrm{Ext}_A^{s,n+s}(Z_2, Z_2) \longrightarrow \pi_n S^0.$$

We will index the Steenrod operations in Ext so that

$$Sq^i\colon \mathrm{Ext}^{s,t} \longrightarrow \mathrm{Ext}^{s+i,2t}.$$

To state the general formula for differentials, the following convention is convenient. If the elements a, b_1 and b_2 are in filtrations s, $s + r_1$ and $s + r_2$, respectively, then

$$d_* a = b_1 \dot{+} b_2$$

means

$$\begin{array}{lll} d_{r_1} = b_1 & \text{if} & r_1 < r_2, \\ d_r = b_1 + b_2 & \text{if} & r = r_1 = r_2, \\ d_{r_2} = b_2 & \text{if} & r_1 > r_2. \end{array}$$

We also need a function which occurs in the study of reducibility of stunted projective spaces and of vector fields on spheres.

Definition 2.1. $v(n) = 8a + 2^b$ if the exponent of 2 in the prime factorization of $n + 1$ is $4a + b$ with $0 \leq b \leq 3$.

If $v = v(n)$ then the attaching map of the n cell of RP^n factors through RP^{n-v} but not through PR^{n-v-1}.

The following result was first proved by Makinen [7]. The author has generalized it to the Adams spectral sequence

$$\mathrm{Ext}_{E_*E}^{s,n+s}(E_*S, E_*Y) \Longrightarrow \pi_n Y$$

for any H_∞ ring spectrum Y and ring spectrum E for which E_*E is π_*E flat [2].

THEOREM 2.2. *Let* $x \in E_r^{s,n}$. Then

$$d_*Sq^i x = Sq^{i+r-1}d_r x \dot{+} \begin{cases} 0 & v > k+1 \text{ or } 2r-2 < v < k \\ axd_r\, x & v = k+1 \\ aSq^{i+v}x & v = k \text{ or } (v < k \text{ and } v \le 10) \end{cases}$$

where $k = s - i$, $v = v(n+k)$ *and* $a \in E_\infty^{\phi, v-1}$ *detects a generator of* $Im\ J$ *in* $\pi_{v-1}S^0$.

The contrast between Steenrod operations and Massey products is instructive. (We mean matric Massey products, of course.) Every element of Ext_A^s, $s > 1$, is decomposable as a Massey product, typically in many different ways. Decomposability in terms of the Steenrod operations is comparatively rare. On the other hand, Massey products have indeterminacy, a complication which Steenrod operations do not share. Finally, there are formulas analogous to Theorem 2.2 for differentials on Massey products; however there are rather stringent conditions which must hold before they apply. In practice, Massey products and Steenrod operations seem complementary, each answering questions the other finds difficult.

§3. THE ELEMENTS IN QUESTION

Table 1 contains the elements of π_*S^0 and of $\mathrm{Ext}_A(Z_2, Z_2)$ with which we shall be concerned. The names for elements of π_*S^0 are those used by Toda[16] with three exceptions. Our η_4 is called η^* in [16] and η_3 in [1], but has more recently been called η_4[5]. The elements θ_4 and t are beyond the range of Toda's calculations, but are unambiguous since π_{30} and π_{36} are each Z_2.

THEOREM 3.1. *The following products are zero in* π_*S^0: (i) $2\eta_4$, $\nu\eta_4$, $2\sigma^2$, $\eta\sigma^2$, $\nu\sigma$. (ii) $\rho\eta_4$, η_4^2. (iii) $\eta^2\theta_4$, $\sigma^2\eta_4$, $2\bar{\sigma}$, $\sigma\bar{\sigma}$.

Proof. (i) and (ii) may be found in [16] and [1], respectively. In (iii) $\sigma^2\eta_4 = 0$ for filtration reasons, while $\sigma^2\theta_4 \in \eta\langle 2, \sigma^2, \eta_4\rangle = \{0\}$, $2\bar{\sigma} \in \langle \nu, \eta, \sigma^2\rangle 2 = \{0\}$ and $\sigma\bar{\sigma} = \sigma\langle \nu, \eta, \sigma^2\rangle = \{0\}$ by the next theorem.

THEOREM 3.2. (i) $\bar{\nu} = \langle \nu, \eta, \nu\rangle$ mod 0. (ii) $\bar{\sigma} = \langle \nu, \eta, \sigma^2\rangle$ mod 0. (iii) $\eta_4 \in \langle \eta, 2, \sigma^2\rangle = \langle \eta, \sigma^2, 2\rangle$ mod $\eta\rho$. (iv) $\eta\theta_4 = \langle \sigma^2, 2, \eta_4\rangle$ mod 0, $\eta\theta_4 \in \langle 2, \sigma^2, \eta_4\rangle$ mod $2J_{31}$.

Proof. (i) and (ii) may be found in [16], while (iii) and (iv) are in [1]. Note that (iii) and (iv) both follow from the differential $d_2h_4 = h_0h_3^2$ by Moss' convergence theorem [12].

THEOREM 3.3. *In* $\mathrm{Ext}_A(Z_2, Z_2)$ (i) $h_1h_3 = \langle h_2, h_1, h_2\rangle$ mod 0. (ii) $e_0 = \langle h_2, c_0, h_2, h_1\rangle$ mod 0, $e_1 = \langle h_3, c_1, h_3, h_2\rangle$. (iii) $f_0 = \langle h_0^2, h_3^2, h_2\rangle$ mod $h_0^2h_2h_4$. (iv) $c_1 = \langle h_2, h_1, h_3^2\rangle$ mod 0. (v) $e_0 = \langle h_0, h_1, h_0^2, h_3^2\rangle$ mod 0, $e_1 = \langle h_1, h_2, h_1^2, h_4^2\rangle$.

Proof. All but (v) follow from the May spectral sequence [10] via results in [9]. (v) occurs in [15] where it is attributed to Zachariou. We give a quick proof due to Mahowald here.

Table 1. Some elements of π_* and of Ext

n	element of π_n	corresponding element of $\mathrm{Ext}^{s,n+s}$	s
1	η	h_1	1
3	ν	h_2	1
7	σ	h_3	1
8	$\bar{\nu}$	h_1h_3	2
14	κ	d_0	4
15	ρ	$h_0^3h_4$	4
16	η_4 (= η^*)	h_1h_4	2
19	$\bar{\sigma}$	c_1	3
20	$\bar{\kappa}$	g	4
30	θ_4	h_4^2	2
31	$\langle\nu, \sigma, \bar{\kappa}\rangle$	n	5
31	J_{31}	$h_0^{10}h_5$	11
32	unspecified	d_1	4
36	t	t	6
37	$\sigma\theta_4$	x	5
44	unspecified	g_2	4
17	not	e_0	4
18	in	f_0	4
38	E_∞	e_1	4

Since e_0 is the only nonzero element in its bidegree, and $h_1e_0 = h_0f_0$, it suffices to note that

$$\begin{aligned} h_1\langle h_0, h_1, h_0^2, h_3^2\rangle &= \langle\langle h_1, h_0, h_1\rangle, h_0^2, h_3^2\rangle \\ &= \langle h_0h_2, h_0^2, h_3^2\rangle \\ &= h_0\langle h_2, h_0^2h_3, h_3\rangle \\ &= h_0\langle h_0^2h_3, h_3, h_2\rangle + h_0\langle h_3, h_2, h_0^2h_3\rangle \\ &= h_0\langle h_0^2, h_3^2, h_2\rangle + h_0^3h_2h_4 \\ &= h_0f_0. \end{aligned}$$

THEOREM 3.4.

$$Sq^id_0 = \begin{cases} d_0^2 \\ 0 \\ r \\ 0 \\ d_1 \end{cases} \quad \textit{and} \quad Sq^ie_0 = \begin{cases} e_0^2 \\ m \\ t \\ x \\ e_1 \end{cases} \quad \textit{if} \quad i = \begin{cases} 4 \\ 3 \\ 2 \\ 1 \\ 0 \end{cases}$$

Proof. This can be found in Milgram [11] or Mukohda [13].

Note. We will have no occasion to use the elements m and r here, so have omitted them from Table 1.

THEOREM 3.5. $d_2e_0 = h_1^2d_0$

Proof. This may be found in [8]. Alternatively, it follows from the algebra structure of Ext and Theorem 2.2 applied to c_0.

§4. NEW RESULTS

We begin with the result from which all the others will follow.

THEOREM 4.1. $d_3e_1 = h_1t = h_2^2n$

Proof. By Theorems 2.2 and 3.5,

$$d_*e_1 = d_*Sq^0e_0 = Sq^1(h_1^2d_0) \dot{+} h_1Sq^2e_0 = 0 + h_1t = h_1t.$$

Note that $\dot{+}$ means $+$ here because both terms are in filtration 7. The relation $h_1t = h_2^2n$ can be found in [14].

Theorem 4.1 corrects Theorem 8.6.6 and Corollary 8.6.4 of [6].

COROLLARY 4.2. (i) $\pi_{38}S^0 = Z_4 + Z_2$, *generated by* $\{h_0^2h_3h_5\}$ *and* $\nu^2\{d_1\} = \{h_2^2d_1\}$ *respectively.* (ii) $\pi_{37}S^0 = Z_2^2$, *generated by* $\{h_2^2h_5\}$ *and* $\{x\}$. (iii) $\eta t = \nu^2\{n\} = 0$.

Proof. This is immediate from [6] and [1] as amended by Theorem 4.1.

This corollary corrects Theorem 1.1.1 and Proposition 7.3.5 of [6] and §4 of [1]. For completeness we include as Table 2 a list of the groups π_nS^0, $29 \le n \le 45$. This replaces Table 1.1.2 of [6], incorporating the changes required by [1] and Corollary 4.2. The homotopy groups given in §4 of [1] are correct once ηt and e_1 are removed from π_{37} and π_{38}, respectively.

COROLLARY 4.3. (i) $\langle\sigma, \bar{\sigma}, \sigma\rangle = \nu\{n\}$. (ii) $\langle\nu, \eta^2, \theta_4\rangle = t \bmod 0$. (iii) $\langle\nu, \eta_4, \eta_4\rangle = t \bmod 0$. (iv) $\langle\bar{\sigma}, 2, \eta_4\rangle = 0 \bmod 0$.

Proof. We will show (i) and (ii) by the Leibniz formula for Massey products ([9], Theorem 4.5), as extended to Toda brackets and the Adams spectral sequence by Kochman [4]. The theorem as stated in [4] or [9] contains a number of technical hypotheses which guarantee that there will be no "interference" preventing the differential from taking the desired form. In both (i) and (ii) one of these hypotheses fails and we must verify by hand that such interference does not occur. Specifically, for (ii) we must show that there is a nullhomotopy of $\langle\eta, \nu, \eta^2\rangle$ in filtration 2 modulo filtration 5. The corresponding Massey

Table 2

n	π_nS^0	n	π_nS^0
29	0	38	4+2
30	2	39	$16+2^5$
31	$64+2^2$	40	$4+2^5$
32	2^4	41	2^5
33	2^5	42	$8+2^2$
34	$4+2^3$	43	8
35	$8+2^2$	44	8
36	2	45	$16+2^3$
37	2^2		

product $\langle h_1, h_2, h_1^2\rangle$ is zero in E_2 but this only gives us a nullhomotopy in filtration 2 modulo filtration 4. We must avoid the possibility that $\langle \eta, \nu, \eta^2\rangle$ will show up as a nonzero element of E_2 in filtration 4. Since $\langle \eta, \nu, \eta\rangle = \nu^2$, we have $\langle \eta, \nu, \eta^2\rangle = \eta\nu^2 = 0$. If we use a nullhomotopy of $\eta\nu$ composed with ν as our nullhomotopy of $\langle \eta, \nu, \eta^2\rangle$, there is no problem. Similarly, in (i) we must show that there is a nullhomotopy of $\langle \bar{\sigma}, \sigma, \nu\rangle$ in filtration 3 modulo filtration 6. We can accomplish this by showing that there are nullhomotopies of c_1h_3 in filtration 3 modulo filtration 5 and of h_3h_2 in filtration 1 modulo filtration 3. The Leibniz formula then says

$$d_3e_1 = d_3\langle h_3, c_1, h_3, h_2\rangle = Y_1h_2 + h_3Y_2$$

where Y_1 detects $\langle \sigma, \bar{\sigma}, \sigma\rangle$ and Y_2 detects $\langle \bar{\sigma}, \sigma, \nu\rangle$. Since $\langle \bar{\sigma}, \sigma, \nu\rangle = 0$ for filtration reasons, $Y_2 = 0$. Theorem 4.1 then implies $Y_1 = h_2n$, establishing (i). Similarly,

$$d_3e_1 = d_3\langle h_1, h_2, h_1^2, h_4^2\rangle = Y_1h_4^2 + h_1Y_2$$

where Y_1 detects $\langle \eta, \nu, \eta^2\rangle$ and Y_2 detects $\langle \nu, \eta^2, \theta_4\rangle$. Since $Y_1 = 0$, it follows that $Y_2 = t$, establishing (ii).

We would prove (iii) by the same technique applied to the Massey product $\langle h_1, h_2, h_1h_4, h_1h_4\rangle$ if this Massey product could be formed. However, $\langle h_1, h_2, h_1h_4\rangle = h_2^2h_4 = h_3^3 \neq 0$, so it cannot be formed. Instead, we use the Jacobi identity and Theorem 3.2 to conclude that

$$\begin{aligned} 0 &\in \langle\langle \nu, \eta, 2\rangle, \sigma^2, \eta_4\rangle + \langle \nu, \langle \eta, 2, \sigma^2\rangle, \eta_4\rangle + \langle \nu, \eta, \langle 2, \sigma^2, \eta_4\rangle\rangle \\ &= \langle \nu, \eta_4, \eta_4\rangle + \langle \nu, \eta, \eta\theta_4\rangle \\ &= \langle \nu, \eta_4, \eta_4\rangle + \langle \nu, \nu^2, \theta_4\rangle, \end{aligned}$$

checking that the indeterminacies of the inner brackets disappear since $\langle \nu, \eta\rho, \eta_4\rangle = \langle \nu, \eta, \rho\eta_4\rangle = 0$ by Theorem 3.1, and $\langle \nu, \eta, 2J_{31}\rangle = 0J_{31} = 0$ since $\pi_5 = 0$. This proves (ii).

For (iv) we again use the Jacobi identity and Theorems 3.1 and 3.2 to conclude that

$$\begin{aligned} 0 &\in \langle\langle \nu, \eta, \sigma^2\rangle, 2, \eta_4\rangle + \langle \nu, \langle \eta, \sigma^2, 2\rangle, \eta_4\rangle + \langle \nu, \eta, \langle \sigma^2, 2, \eta_4\rangle\rangle \\ &= \langle \bar{\sigma}, 2, \eta_4\rangle + \langle \nu, \eta_4, \eta_4\rangle + \langle \nu, \eta, \eta\theta_4\rangle = \langle \bar{\sigma}, 2, \eta_4\rangle. \end{aligned}$$

The indeterminacies in (ii)–(iv) are zero because all products in π_{36} are 0.

Corollary 4.3 corrects the presumption in [1] that the brackets in (i)–(iii) are zero. (They would have to be 0 if $\nu^2\{n\} = \eta t \neq 0$.) The bracket in (iv) is new.

§5. NEW PROOFS

The differentials $d_3(h_3h_5) = 0$ and $d_4(h_3h_5) = h_0x$ in §7 of [6] were proved using the false Proposition 7.3.5 (which stated that π_{37} had three generators). However, these differentials are forced by the other differentials involving the 37 stem and the fact that $\sigma\theta_4 \neq 0$. Specifically, the differentials $d_2P^1k = h_0P^2g$ ([6], 1.1.5), $d_2y = h_0^3x$ ([6], 5.1.4), $d_3e_1 = h_1t$ (Theorem 4.1) and $d_4e_0g = P^2g$ ([6], 4.2.1) imply that if $d_3(h_3h_5) \neq 0$ then $\pi_{37} = Z_2$, generated by $\{h_2^2h_5\}$. This would imply that $\sigma\theta_4 = 0$, contradicting ([6], 7.3.2). Thus $d_3(h_3h_5) = 0$.

To see that $d_4(h_3h_5) = h_0x$ we need only show that $\sigma\theta_4$ is detected by x, since $2\theta_4 = 0$ then forces h_0x to be killed by something, and h_3h_5 is the only candidate. In Ext for the cofiber of σ there is a differential $d_3(\overline{h_4^2}) = x$ ([6], Lemma 7.3.1), where $\overline{h_4^2}$ projects to h_4^2 on the top cell. It then follows by a standard lemma about homotopy exact couples that x detects the composite of σ and θ_4 (see [3], Theorem 1.24 for a proof in the case of Adams spectral sequences for cohomotopy).

Proposition 3.1.5 of [1] gives the Toda brackets

$$\langle \bar{\nu}, \sigma, \bar{\kappa} \rangle = \langle \bar{n}, \eta, \nu \rangle = t \bmod 0,$$

where $\bar{n} = \langle \nu, \sigma, \bar{\kappa} \rangle$. The claim that $\eta t \neq 0$ was used to show the indeterminacies are 0 and to establish the first bracket. The second bracket follows from the relation $h_2{}^2 n = h_1 t$ as in [1]. With the correct relation $\eta t = 0$, it is still easy to verify that all products in π_{36} are 0; the only nontrivial case, $\nu\{p\} = 0$, following from the bracket $\langle \eta_4, \eta_4, 2 \rangle \subset \{p\}$ ([1], 3.3.3). (Recall that η_4 is called η_3 in [1].) Thus, all three fold brackets in π_{36} have 0 indeterminacy. The first bracket follows by a manipulation:

$$\begin{aligned}\langle \bar{\nu}, \sigma, \bar{\kappa} \rangle &= \langle \langle \nu, \eta, \nu, \rangle, \sigma, \bar{\kappa} \rangle \\ &= \langle \nu, \eta, \langle \nu, \sigma, \bar{\kappa} \rangle \rangle \\ &= \langle \nu, \eta, \bar{n} \rangle\end{aligned}$$

using $\bar{\nu} = \langle \nu, \eta, \nu \rangle$ and $\langle \eta, \nu, \sigma \rangle \in \pi_{12} = 0$ from [16].

Another change required in [1] is the elimination of Proposition 3.5.4. which asserts that $\langle \eta_4, \eta_4, \nu, \sigma \rangle = \{g_2\}$. In fact, Corollary 4.3 implies that the four fold bracket cannot be constructed.

Finally, $\eta\{g_2\}$ cannot be decomposed as $\sigma\{e_1\}$ as claimed in §4 of [1], since e_1 is not a permanent cycle. Similarly, there is no extension question between e_1 and $h_2{}^2 d_1$, eliminating the need for Part 1 of [15].

The author knows of no other significant changes in [1], [6] or [15] forced by the new differential.

Acknowledgements— I am indebted to Mark Mahowald and Peter May for numerous helpful conversations and in particular for explaining the Toda bracket implications to me.

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